

## A RELATIVISTIC VERSION OF THE GAUSS-BONNET FORMULA

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### Introduction

The Gauss-Bonnet formula relates the sum of the exterior angles of a geodesic polygon on a surface to the total Gaussian curvature which the polygon encloses. Thus one obtains such statements as: the sum of the interior angles of a geodesic triangle is  $\pi$  if and only if the total curvature enclosed by the triangle is zero.

To develop a version of the formula which applies to surfaces with an indefinite metric requires only a careful definition of a quantity to replace "angle" and a check that the arguments of the definite case remain valid. This is done in §§ 1 and 2.

In § 3 an example is given to indicate the kind of physical quantity which the total Gaussian curvature might measure.

### 1. The flat case

In this section the "pseudo-angle" or "proper velocity" between two vectors in a plane with indefinite metric is defined and some elementary properties listed.

Let  $M^2$  denote the space of pairs of real numbers with inner product

$$(1) \quad \langle (a_1, a_2), (b_1, b_2) \rangle = -a_1 a_2 + b_1 b_2 .$$

Take the positive orientation of  $M^2$  to be that given by the vector space basis  $\{e_1 = (1, 0), e_2 = (0, 1)\}$ .

Let  $\alpha: I \rightarrow M^2$  be a continuously differentiable curve parametrized with respect to proper time, i.e.,

$$(2) \quad \langle \alpha'(s), \alpha'(s) \rangle = -1, 1, 0 .$$

The curve  $\alpha$  is called timelike, spacelike or null respectively.

Next define a moving frame  $\{T(s), N(s)\}$  on  $\alpha$  as follows. Let  $\{u_1, u_2\}$  be an orthonormal frame at  $\alpha(s)$ , and set

$$T(s) = \alpha'(s) = x_1 u_1 + x_2 u_2 ,$$

$$(3) \quad N(s) = \begin{cases} x_2 u_1 + x_1 u_2 & \text{if } \{u_1, u_2\} \text{ has positive orientation ,} \\ -(x_2 u_1 + x_1 u_2) & \text{if } \{u_1, u_2\} \text{ has negative orientation .} \end{cases}$$

The definition of  $N$  is independent of the choice of  $\{u_1, u_2\}$  since  $N$  is simply  $T$  reflected in one piece of the light cone.

Lastly define a real valued function  $\phi$  with domain  $I$  by

$$(4) \quad \phi(s) = \begin{cases} \ln |a + b| & \text{if } a + b \neq 0 , \\ -\ln |a - b| & \text{if } a + b = 0 , \end{cases}$$

where  $T(s) = ae_1 + be_2$ . Since  $|a + b||a - b| = 1$  or  $0$ , the two functions on the right hand side of (4) are equal where they are both defined.

**Theorem 1.** *There is a unique function  $g$  defined on  $I$  for which*

$$T'(s) = g(s)N(s) , \quad N'(s) = g(s)T(s) .$$

In fact  $g = \phi'(s)$ .

*Proof.* Since  $\alpha$  is parameterized with respect to proper time, using the logarithmic forms of the inverse hyperbolic functions one sees that  $T$  may be written in one of the forms:

$$\pm(e_1 \cosh \phi + e_2 \sinh \phi) , \quad \pm(e_1 \sinh \phi + e_2 \cosh \phi) , \quad \pm a(e_1 \pm e_2) .$$

Direct calculation now gives the theorem. q.e.d.

The Euclidean version of Theorem 1 is the starting point of the theory of plane curves. There  $s$  is the arc length and  $\phi$  is the angle which  $T$  makes with the  $x$ -axis. Here  $\phi$  is the "pseudo-angle" which  $T$  makes with  $e_1$ , i.e., with the time axis. The functions  $T, N, g$  are invariants in the sense that their definition does not depend on the choice of basis  $\{e_1, e_2\}$ . On the other hand if one changes, the basis  $\phi$  will change by an additive constant and its sign depends on the orientation of the basis. As in the Euclidean theory, it may be shown that  $g$  determines  $\alpha$  up to a Lorentz transformation (translations included).

Suppose a particle is constrained to move in one spatial dimension, say the  $e_2$  axis where  $\{e_1, e_2, e_3, e_4\}$  is an orthonormal basis of the Minkowski space of special relativity ( $c = 1$ ). Then by suppressing the irrelevant directions  $e_3, e_4$  we may consider the space time trace of the particle to be the curve  $\alpha$  above. In this case  $g(s)$  is the acceleration at time  $s$  as measured by an observer at rest with respect to the particle and since  $\phi' = g$ , one might call  $\phi$  a "proper velocity."

For an observer at rest with respect to the frame  $\{e_1, e_2\}$  the expression

$$\alpha(t) = te_1 + x(t)e_2$$

describes the motion of the particle. Taking  $\{e_1, e_2\}$  oriented so that  $T = e_1 \cosh \phi + e_2 \sinh \phi$  we see that  $t'(s) = \cosh \phi$  and hence the speed of the particle measured by this observer is

$$v = \frac{dx}{dt} = \frac{ds}{dt} \frac{dx}{ds} = \tanh \phi ,$$

and so  $\phi = \tanh^{-1} v = v + \frac{1}{3}v^3 + \dots$ . Thus for  $v \ll 1$ ,  $\phi$  is indistinguishable from  $v$ . The sum formula for the hyperbolic tangent shows that composing velocities corresponds to adding  $\phi$ 's.

For a particle moving with the speed of light,  $\phi$  is the logarithm of twice the energy ( $= e_1$  component of  $T$ ). This reduces to

$$(5) \quad \phi = \log v + \text{const} ,$$

where  $\nu$  is the frequency and hence  $g = \phi' = \nu'/\nu$ . If  $\alpha$  is spacelike,  $\phi$  gives the relative velocity of the orthonormal frame  $\{N, T\}$  with respect to  $\{e_1, e_2\}$ .

To define an "angle" between any two unit or null vectors proceed as follows. If  $\mathcal{O} = \{u_1, u_2\}$  is an orthonormal basis, and  $u = au_1 + bu_2$  is a unit or null vector, then define  $\phi_{\mathcal{O}}(u)$  by (4). If  $\mathcal{O}' = \{u'_1, u'_2\}$ , it is not difficult to verify the formulas

$$(6) \quad \begin{aligned} \phi_{\mathcal{O}'}(u) &= \phi_{\mathcal{O}}(u) + \phi_{\mathcal{O}'}(u_1) && \text{if } \mathcal{O} \text{ and } \mathcal{O}' \text{ similarly oriented ,} \\ -\phi_{\mathcal{O}'}(u) &= \phi_{\mathcal{O}}(u) - \phi_{\mathcal{O}'}(u_1) && \text{if } \mathcal{O} \text{ and } \mathcal{O}' \text{ oppositely oriented .} \end{aligned}$$

If  $u, v$  are unit or null vectors, and  $\mathcal{O}$  is an orthonormal set, define  $\phi_{\mathcal{O}}(u, v) = \phi_{\mathcal{O}}(u) - \phi_{\mathcal{O}}(v)$ . It follows from (6) that  $\phi_{\mathcal{O}}(u, v)$  depends only on the orientation of  $\mathcal{O}$ . Thus define  $\phi(u, v) = \phi_{\mathcal{O}}(u, v)$  where  $\mathcal{O}$  is any positively oriented orthonormal basis of  $M^2$ . If  $u_1, \dots, u_n$  are unit or null vectors we have the following formulas

$$(7) \quad \phi(u_1, u_2) = -\phi(u_2, u_1) ,$$

$$(8) \quad \phi(u_1, u_2) + \phi(u_2, u_3) = \phi(u_1, u_3) ,$$

$$(9) \quad \phi(u_1, u_2) + \dots + \phi(u_{n-1}, u_n) + \phi(u_n, u_1) = 0 .$$

Formula (9) is the simplest case of the Gauss-Bonnet theorem. The corresponding statement in the Euclidean plane is that the exterior angles of a polygon sum to  $2\pi$ .

### 2. General case

Throughout this section  $M$  will denote a Minkowski surface, i.e., an abstract surface with each tangent plane a Minkowski plane. Attention will be

restricted to a region of  $M$  oriented by a frame field  $\{E_1, E_2\}$ . The following notation will be used. A general reference for the Euclidean case is [2, Chapter 7].

The dual 1-forms  $\theta_1, \theta_2$  are defined by  $\theta_i(E_j) = \langle E_i, E_j \rangle$ . The connection forms  $\omega_{ij}$  are defined by the equations

$$d\theta_1 = \omega_{12} \wedge \theta_2, \quad d\theta_2 = \omega_{21} \wedge \theta_1, \quad \omega_{12} = \omega_{21},$$

where  $d$  denotes the exterior derivative, and  $\wedge$  the wedge or exterior product. The "area form" is " $dM$ " =  $\theta_1 \wedge \theta_2$ ; this form depends only on the orientation of  $\{E_1, E_2\}$ . The Gaussian curvature  $K$  is defined by the formula

$$d\omega_{12} = -K\theta_1 \wedge \theta_2.$$

The covariant derivative in the direction of the tangent vector  $v$  is denoted by  $\nabla_v$ . Recall that its action on a vector field  $Y = y_1E_1 + y_2E_2$  is given by

$$(10) \quad \nabla_v Y = (v[y_1] + y_2\omega_{21}(v))E_1 + (v[y_2] + y_1\omega_{12}(v))E_2,$$

where  $v[f]$  denotes the directional derivative of the function  $f$  in the direction  $v$ .

Let  $\alpha: I \rightarrow M$  be a continuously differentiable curve parameterized with respect to proper time. Let  $T(s) = \alpha'(s)$ . Then  $T$  is a unit or null vector field along  $\alpha$ . If  $T(s) = a(s)E_1(\alpha(s)) + b(s)E_2(\alpha(s))$ , set  $N(s) = bE_2 + aE_1$  along  $\alpha$ , and define  $g(s)$  by

$$\nabla_{\alpha'} T = gN.$$

At each point of  $\alpha(s)$  define  $\phi(s)$  by

$$\phi(s) = \phi_{\{E_1, E_2\}}(T(s)).$$

Using (10) we immediately generalize Theorem 1 to

**Theorem 2.** *Let  $\alpha: I \rightarrow M$  be a differentiable curve parameterized with respect to proper time and having image contained in a region oriented by the frame field  $\{E_1, E_2\}$ . Then*

$$g(s) = d\phi/ds + \omega_{12}(\alpha'(s)).$$

Thus the acceleration measured by an observer riding with  $\alpha$  breaks into two parts. The term  $\phi'(s)$  is due to motion relative to the frame field  $\{E_1, E_2\}$ , and the term  $\omega_{12}(\alpha')$  is due to the acceleration in the frame field itself. Notice that  $\alpha$  is a geodesic of  $M$  if and only if  $g \equiv 0$ . If  $M$  is a submanifold of a higher dimensional space, then  $g$  gives that component of acceleration in the larger space which is tangent to  $M$ . The corresponding Euclidean concept is that of *geodesic curvature*.

**Theorem 3 (Gauss-Bonnet formula).** *Let  $R$  be a region in the plane, and  $X: R \rightarrow M$  a restriction of a coordinate patch mapping. Let  $X[R]$  lie in a region oriented by the frame field  $\{E_1, E_2\}$ , and the boundary of  $X[R]$  be given by  $\partial X = \sum_1^n \alpha_i$  where  $\alpha_i: [a_i, b_i] \rightarrow M$  is a continuously differentiable curve parameterized with respect to proper time. Assume  $\alpha_{i+1}(a_{i+1}) = \alpha_i(b_i)$  for  $i \leq n - 1$ , and  $\alpha_1(a_1) = \alpha_n(b_n)$ . Set  $\phi_{i,i+1} = \phi(\alpha'_{i+1}(a_{i+1}), \alpha'_i(b_i))$  for  $i \leq n - 1$ , and  $\phi_{n,1} = \phi(\alpha'_1(a_1), \alpha'_n(b_n))$ . Then*

$$\iint_X KdM + \int_{\partial X} g + \phi_{12} + \dots + \phi_{n-1,n} + \phi_{n,1} = 0 .$$

*Proof.* By Stokes theorem

$$\iint_X d\omega_{12} = \int_{\partial X} \omega_{12} .$$

Since  $d\omega_{12} = -K\theta_1 \wedge \theta_2 = -KdM$  it is sufficient to evaluate

$$\int_{\partial X} \omega_{12} = \sum_1^n \int_{\alpha_i} \omega_{12} .$$

To evaluate a typical term of this integral, apply Theorem 2 to get

$$\begin{aligned} \int_{\alpha_i} \omega_{12} &= \int_{a_i}^{b_i} \omega_{12}(\alpha'_i(s))ds = \int_{a_i}^{b_i} g(s)ds - \int_{a_i}^{b_i} \frac{d\phi}{ds} ds \\ &= \int_{a_i} g + \phi(a_i) - \phi(b_i) . \end{aligned}$$

Since by definition we have  $\phi_{i,i+1} = \phi(a_{i+1}) - \phi(b_i)$  for  $i \leq n - 1$  and  $\phi_{n,1} = \phi(a_1) - \phi(b_n)$ , summing the last formula gives the desired result.

**Remark.** In the notation of § 1,  $\phi_\phi(u) = \phi_\phi(-u)$  for any unit or null vector  $u$ . This means that the direction in which each boundary curve is traced affects only the integral of  $g$  in Theorem 3. If the boundary curves are geodesics of  $M$ , then  $g \equiv 0$  and this integral drops out.

### 3. Example-a Doppler formula

Suppose that a photon is emitted at a point  $A$  in the space-time of general relativity and observed at a point  $B$ . Let  $\alpha_1$  be the space-time trace of the photon from  $A$  to  $B$ , which is assumed to be a geodesic. Let  $\alpha_2$  be the space-time geodesic which the source would follow if unaccelerated. Let  $\beta$  be the space-time trace of the observer. Let  $\alpha_3$  be a spacelike geodesic which

- i) cuts  $\beta$  orthogonally at  $B$  and ii) intersects  $\alpha_2$  at some point  $C$ .

The curve  $\alpha_3$  will exist if the region under consideration lies in a sufficiently small geodesic neighborhood of  $A$ . It is the intersection of two geodesic sub-

manifolds, the first of which is the Euclidean 3-manifold of all geodesics through  $B$  orthogonal to  $\beta$ , i.e., that portion of space-time which the observer calls space at the instant when he observes the photon. The second submanifold is Minkowski surface of all geodesics emanating from  $A$  and tangent to the plane of tangent vectors spanned by  $\alpha'_1$  and  $\alpha'_2$  at  $A$ . Let  $\mathcal{A}$  denote the section of this latter manifold bounded by the curves  $\alpha_i$ . Then Theorem 3 gives the formula

$$(11) \quad \iint_{\mathcal{A}} K dM + \phi_{12} + \phi_{23} + \phi_{31} = 0.$$

By (7),  $\phi_{12} = -\phi_{21}$  and so  $\phi_{12} = -(\ln \nu_e + a)$  by the remarks following Theorem 1 where  $\nu_e$  is the frequency of the emitted photon. By (8), we have

$$\phi_{31} = \phi(\alpha'_3, \alpha'_1) = \phi(\alpha'_3, \beta') + \phi(\beta', \alpha'_1) = \phi(\beta', \alpha') = \ln \nu_a + a,$$

where  $\nu_a$  is the frequency of the photon observed at  $B$ . (Since  $\alpha_3$  is orthogonal to  $\beta$  one finds  $\phi(\beta', \alpha'_3) = \ln 1 = 0$ .)

The pair  $\beta', \alpha'_3$  is a frame at  $B$ . Parallely translate this frame along  $\alpha_3$  to  $C$ . Since  $\alpha_3$  is a geodesic, the resulting frame is of the form  $\{u, \alpha'_3\}$  and it is the frame at  $C$  which is "at rest" with respect to the observer at the event  $B$ . Thus by (7) and (8)

$$\phi_{23} = \phi(\alpha'_2, \alpha'_3) = -\phi(u, \alpha'_2) + \phi(u, \alpha'_3) = -\tanh^{-1} v = \frac{1}{2} \ln \left\{ \frac{1-v}{1+v} \right\},$$

where  $v$  is the velocity of the unaccelerated source with respect to the observer at the moment when the photon is observed.

Substituting these values for the  $\phi_i$  into (11) gives

$$\nu_e = \nu_a \sqrt{\frac{1-v}{1+v}} \exp \left\{ \iint_{\mathcal{A}} K dM \right\}.$$

In the case where  $\mathcal{A}$  is flat ( $K \equiv 0$ ), this reduces to the usual formula from special relativity.

### References

- [1] S. S. Chern, *Differentiable manifolds*, Lecture notes, University of Chicago, 1959.
- [2] B. O'Neill, *Elementary differential geometry*, Academic Press, New York, 1966.

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